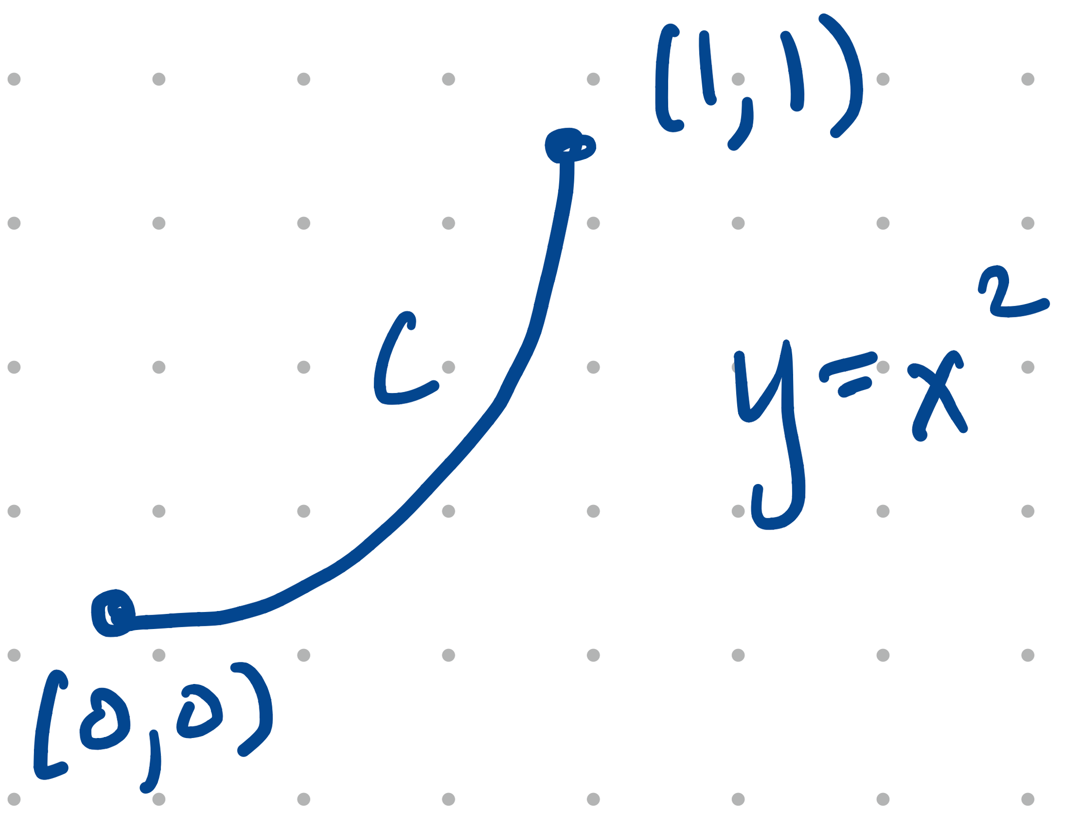
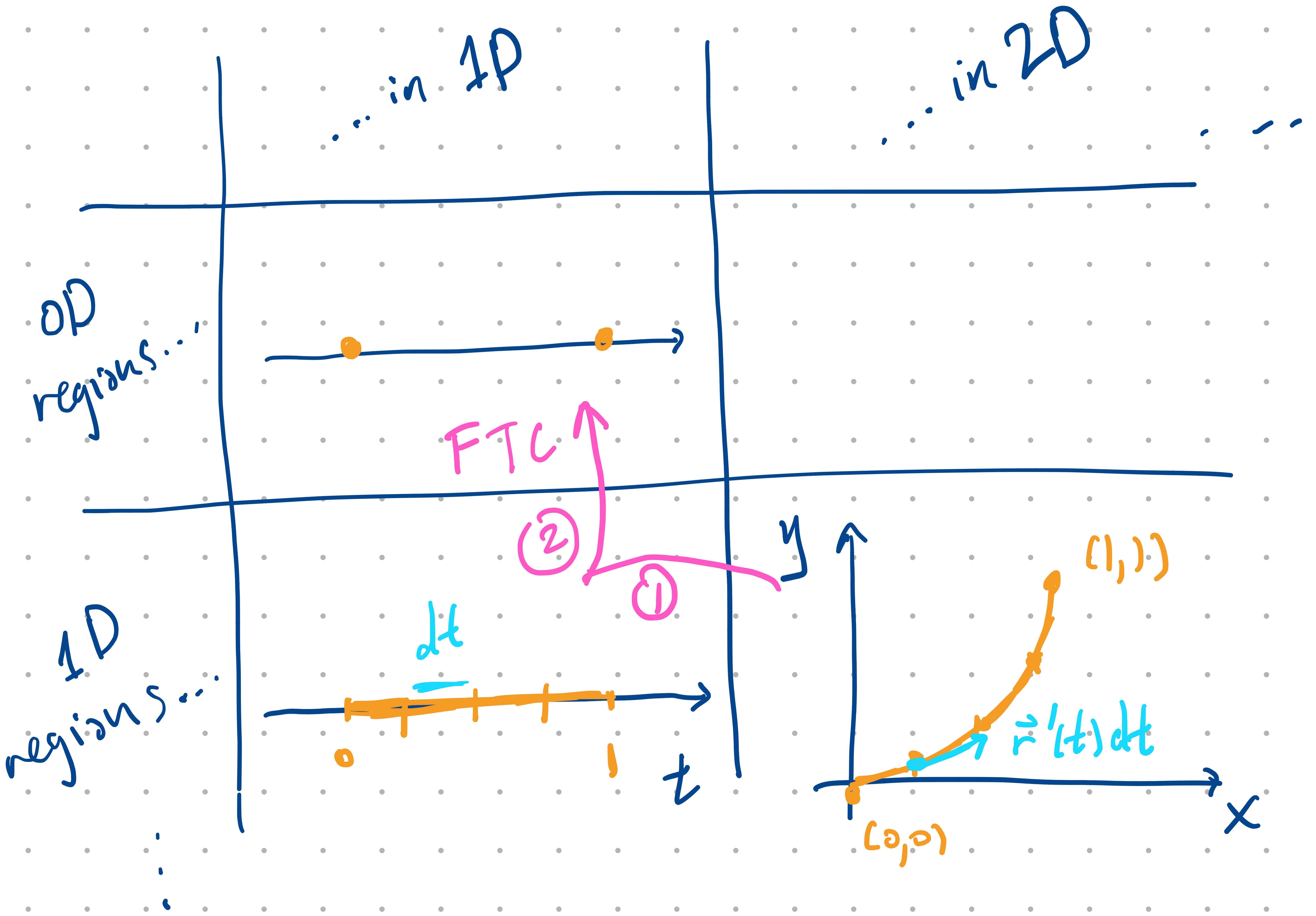
Suppose we have a cable in the shape of



with variable mass density $\sigma(x,y)=\sqrt{y}$ (kg/m)

Its total mass:

$$M=\int_C dm=\int_C \sigma(x,y)\,ds$$



① Parametrize C

Since C is (part of) the graph of $f(x)=x^2$, it has a very natural parametrization where $x=t$,

$$C: \quad \vec{r}(t) = \langle x, y \rangle$$
$$= \langle t, t^2 \rangle \qquad 0 \le t \le 1$$

Using this: (and $ds = |\vec{r}\,'(t)|\,dt$ b/c $\frac{ds}{dt} = |\vec{r}\,'(t)|$ speed = magnitude of vel.)

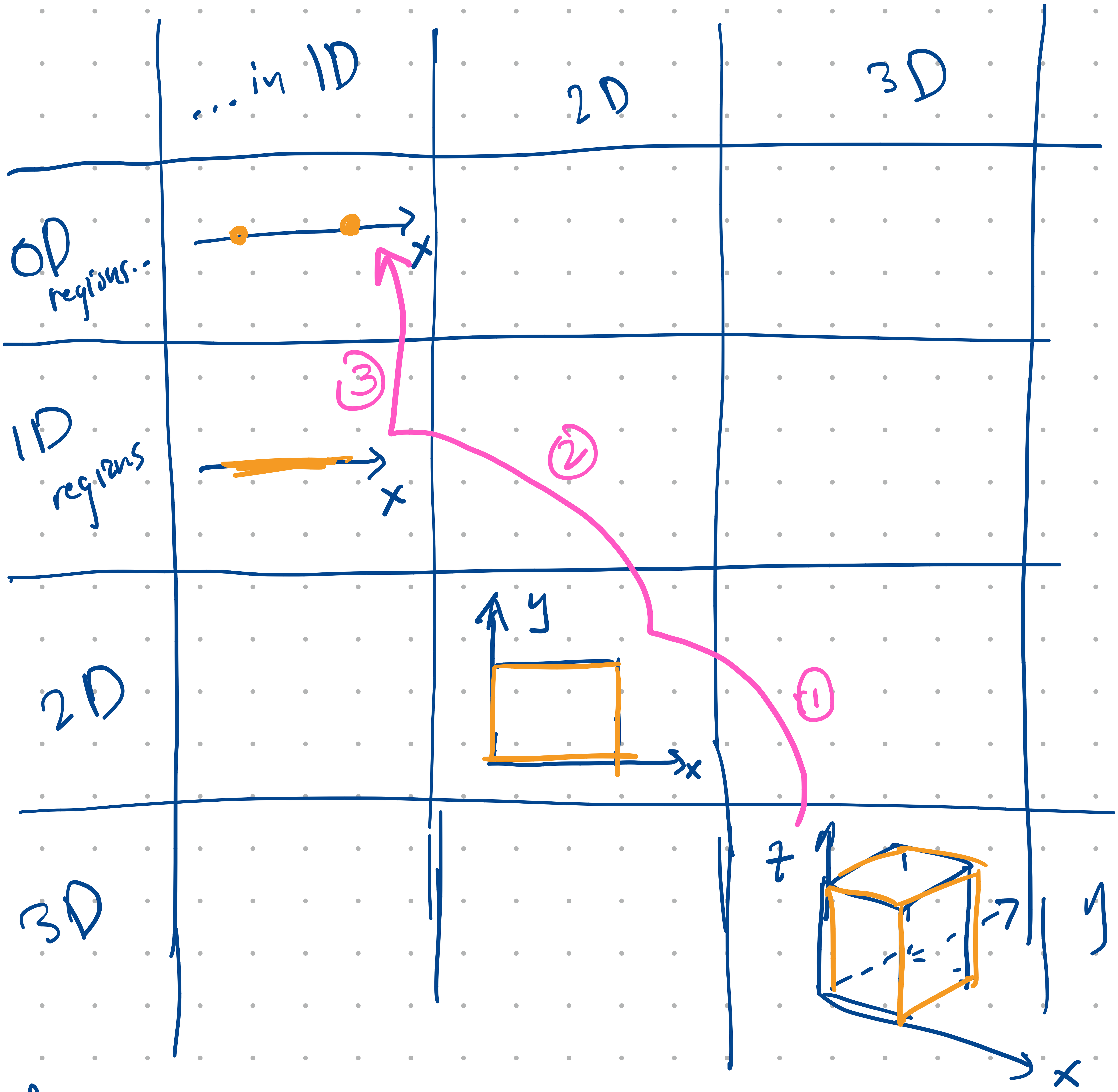
$$\vec{r}\,'(t) = \langle 1, 2t \rangle, \text{ so } |\vec{r}\,'(t)| = \sqrt{1^2 + (2t)^2}.$$

$$\int_C \sqrt{y}\,ds = \int_0^1 \sqrt{t^2} \cdot \sqrt{1+4t^2}\,dt$$

$$= \int_0^1 \frac{1}{8} \overbrace{8t \underbrace{\sqrt{1+4t^2}}_{u}\, dt}^{du} \qquad u = 1+4t^2$$

②

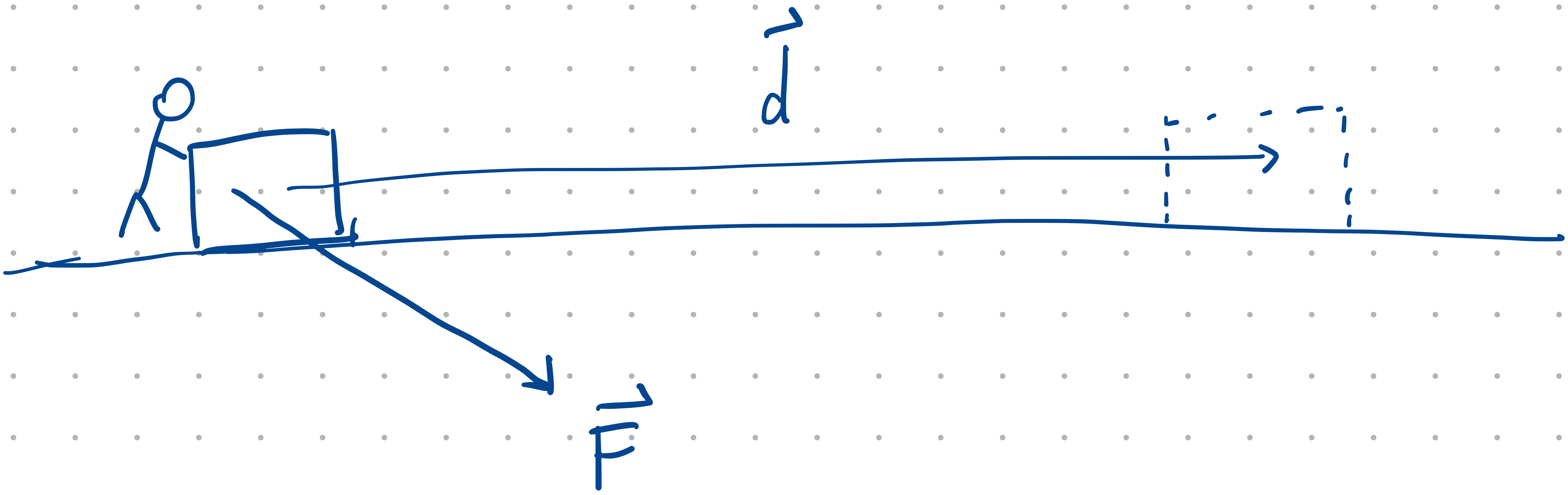
$$= \frac{1}{8} \cdot \frac{2}{3} (1+4t^2)^{3/2} \Big|_{t=0}^{1} = \frac{1}{12} (5^{3/2} - 1) \text{ (kg)}$$



$$\int_e^f \int_c^d \int_a^b \quad dz\,dy\,dx \overset{1}{=} \int_e^f \int_c^d \quad dy\,dx$$

$$\overset{2}{=} \int_e^f \quad dx \overset{3}{=} \left(\quad \right)\Bigg|_{x=e}^{f}$$

The goal is to end up in the first row of the table (because those are the only things we can compute directly).

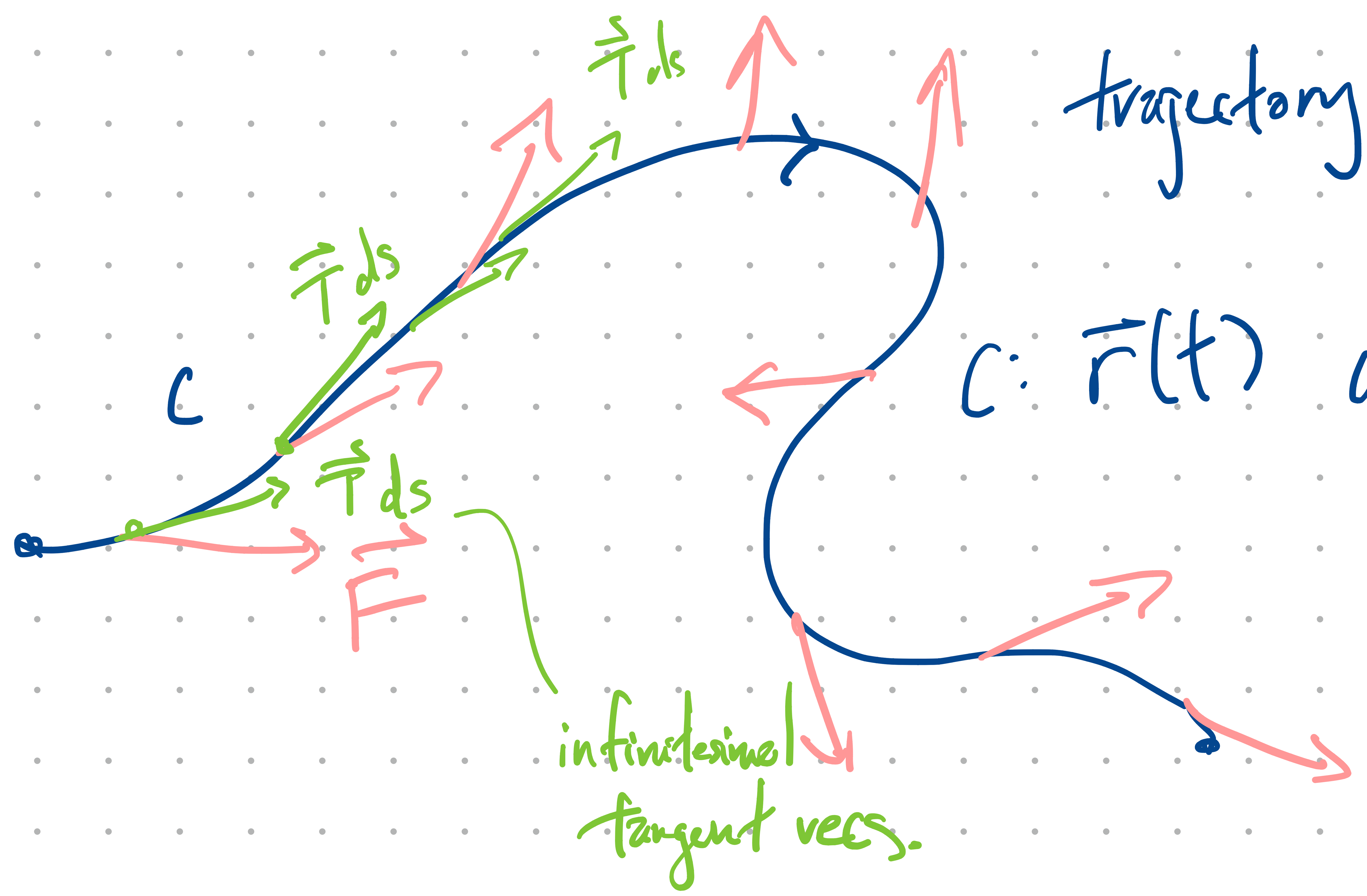


<u>Defn</u>. The work W done by this force is

$$\vec{F}\cdot\vec{d}.$$

(Here we assume $\vec{F}$ is constant).

But more generally, one might have



$C: \vec{r}(t) \quad a \le t \le b$

$$W = \int_C \boxed{\vec{F}\cdot\vec{T}}\, ds = \int_C \vec{F}\cdot d\vec{r}$$

scalar function

$$= \int_a^b \vec{F}\cdot \frac{\vec{r}\,'(t)}{\cancel{|\vec{r}\,'(t)|}}\,\cancel{|\vec{r}\,'(t)|}\,dt = \int_a^b \vec{F}\cdot\vec{r}\,'(t)\,dt$$

(rewritten in terms of t)

$$\int_C f\,ds$$

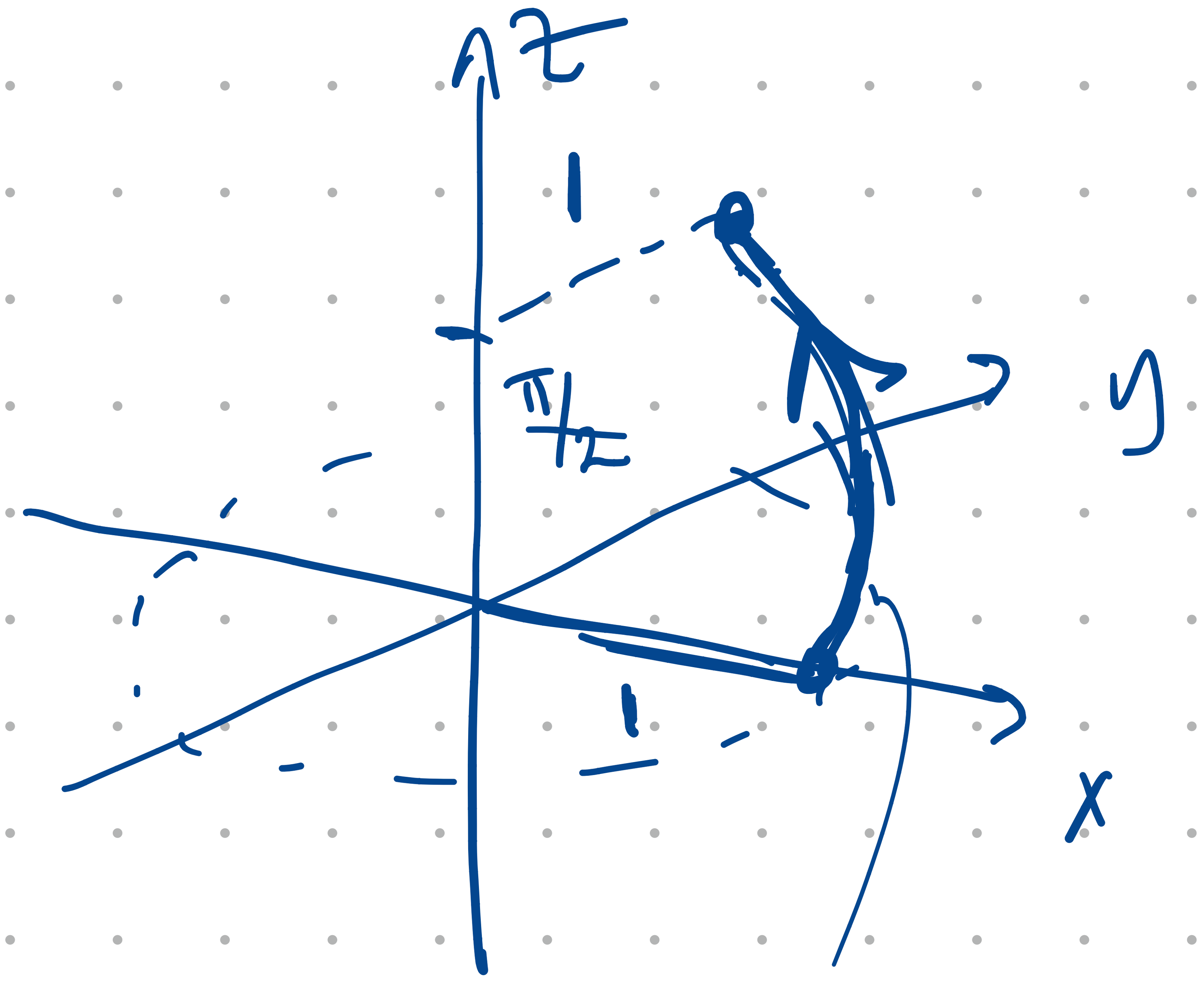
$$\int_C \vec{F}\cdot\vec{T}\,ds = \int_C \vec{F}\cdot d\vec{r}$$

this requires a

choice of orientation of C.

Ex)

$$\int_C \vec{F} \cdot d\vec{r} \quad \text{for}$$



part of a helix

$\vec{F} = \langle x, z, y \rangle$

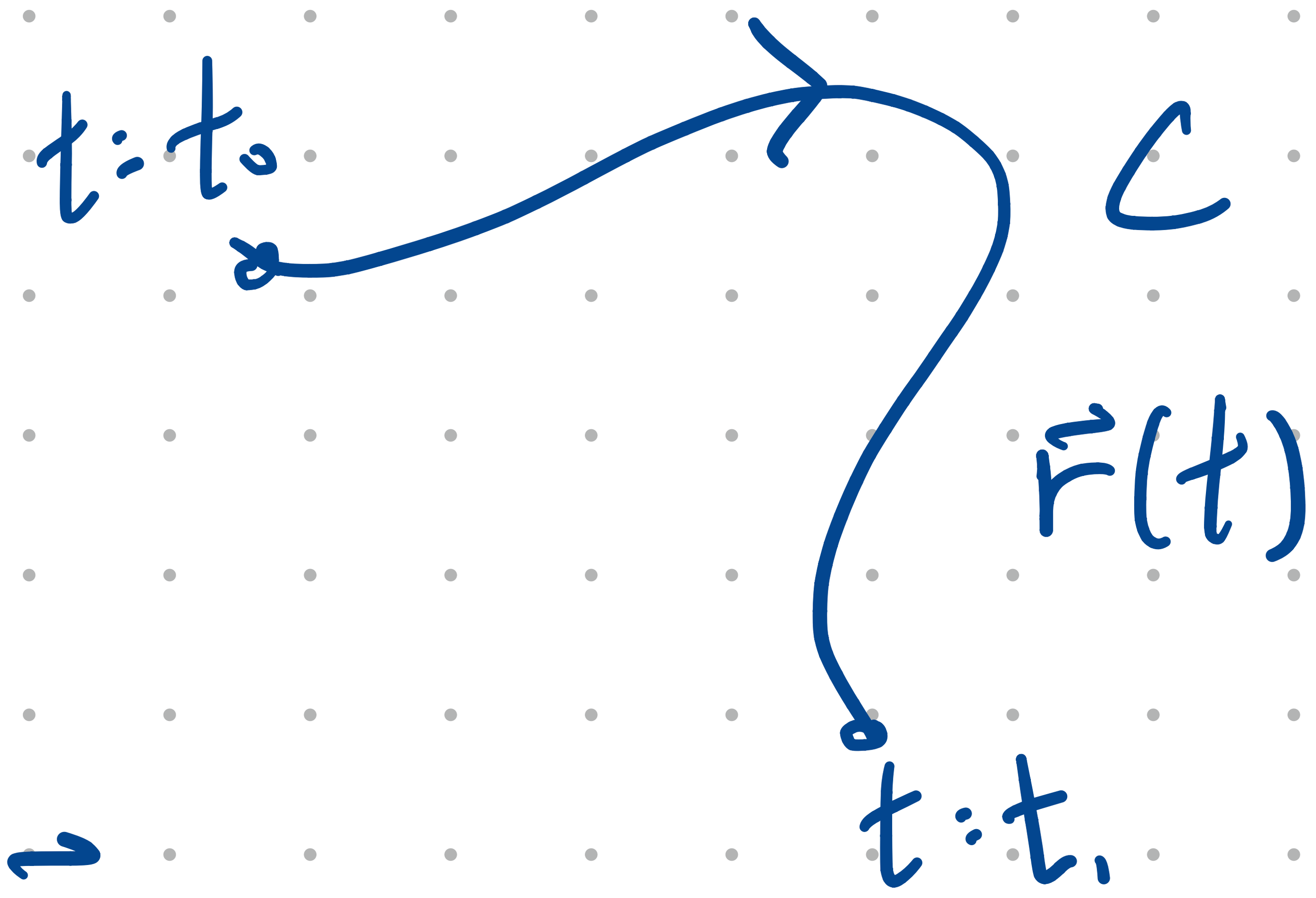
$\vec{r}(t) = \langle \cos t, \sin t, t \rangle$

$$\int_0^{\pi/2} \langle \cos t,\ t,\ \sin t \rangle \cdot \langle -\sin t, \cos t, 1 \rangle \, dt$$

$$= \cdots\cdots = \frac{\pi - 1}{2}$$

	in 1D	2D	3D
0D			
1D	0 $\quad \pi/2 \quad t$		z, y, x

Ex) $\vec{F} = m\vec{a}$



$$W \underset{\text{def}}{=} \int_C \vec{F} \cdot d\vec{r} = \int_C m\vec{a} \cdot d\vec{r}$$

$$= \int_{t_0}^{t_1} m\,\vec{r}\,''(t) \cdot \vec{r}\,'(t)\,dt$$

<u>Exercise</u>: $\vec{F}(t)$, $\vec{G}(t)$

$$(\vec{F} \cdot \vec{G})' = \vec{F}' \cdot \vec{G} + \vec{F} \cdot \vec{G}'$$

scalar fn. of t

(check this!)

In particular:

$$(\vec{r}\,' \cdot \vec{r}\,')' = \vec{r}\,'' \cdot \vec{r}\,' + \vec{r}\,' \cdot \vec{r}\,''$$

$$= 2\vec{r}\,' \cdot \vec{r}\,''$$

So $W = \int_{t_0}^{t_1} m\, \vec{r}\,'' \cdot \vec{r}\,'\, dt$

$$= \frac{1}{2} m (\vec{r}\,' \cdot \vec{r}\,') \Big|_{t=t_0}^{t_1}$$

$$= \frac{1}{2} m v^2 \Big|_{t=t_0}^{t_1} = \Delta \text{ Kinetic Energy}$$

// def

Kinetic energy